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We establish the monotonicity of second-best optimal contracts in the cost-benefit principal-agent model developed by Grossman and Hart (G–H). For a three-state, finite-action version of the model, we prove that the monotone likelihood ratio condition (MLRC) on the probabilities of outcomes guarantees that the optimal payment to the agent is monotonically increasing in output. We estabish a key result that exposes an error in an example that cause G–H to impose conditions stronger than MLRC in order to obtain monotonicity. This result is instrumental in our proof of the monotonicity of second-best optimal incentive schemes.

Key Words: Principal-Agent, Monotonicity, Incentive Schemes, Monotone Likelihood Ratio Condition.

MONOTONICITY OF SECOND-BEST OPTIMAL CONTRACTS

By George E. Monahan and Vijay K. Vemuri

1. INTRODUCTION

In a seminal paper, Grossman and Hart (hereafter denoted G–H)(1983) introduce the cost-benefit version of the principal-agent model and derive many interesting properties of optimal sharing rules. Some of their analysis relates to the monotonicity of second-best optimal sharing rules. In "first-order" principal-agent models, a "standard" condition guaranteeing that the optimal payment to the agent is monotonically increasing in output is the Monotone Likelihood Ratio Condition (MLRC). MLRC places restrictions on the probabilities of outcomes specified as functions of the agent's action. In first-order models, the agents choose from a continuum of actions. To assure uniqueness of the action that solves the agent's incentive compatibility condition, a second requirement, called the Convex Distribution Function Condition (CDFC), is also placed on the output probabilities. See Rogerson (1985) for a discussion of first-order agency models. Milgrom (1981) discusses the general issues associated with monotonicity in contractual settings.

Using a three-state, three-action numerical example, G–H claim to demonstrate that MLRC alone is not sufficient for monotonicity in their cost-benefit model. To establish monotonicity, they invoke fairly strong conditions.

In this paper, we show that the G–H example contains a flaw that reopens the issue of the sufficiency of MLRC. Could a more judicious choice of parameter values rectify the example? One contribution of this paper is to prove that in the three-state version of the cost-benefit model, the answer to this question is no. The result we use to investigate the numerical example also plays a pivotal role in the proof that MLRC alone is indeed sufficient for monotonicity of second-best optimal contracts.

Our analysis is done entirely within the general cost-benefit framework of G–H. We assume, however, that there are only three underlying states of the world. We do permit any finite number of actions are available to the agent. Even with the restriction to three

states, the monotonicity result is surprisingly difficult to establish. Indeed, the entire paper is devoted to this single task. Our result suggests that MLRC is sufficient for monotonicity in the general, finite-state cost-benefit model. We are currently exploring ways in which the results established here can be used to prove this conjecture.

The paper is organized as follows. We briefly review the cost-benefit model and introduce the bulk of the notation in the next section. In Section 3, we discuss MLRC and the G–H numerical example. Next, we present and prove a result that illuminates the fallacy in the G–H example. Section 5 contains preliminary results related to monotonicity. Finally, Section 6 contains the proof that in the finite-action, three-state cost- benefit model, MLRC does indeed imply the monotonicity of second-best optimal contracts. A tedious proof of an intermediate result is relegated to an Appendix.

2. THE COST-BENEFIT PRINCIPAL-AGENT MODEL

We briefly review the cost-benefit model, adopting the notation of G–H (1983). See G–H for complete details.

Notation:

$q_i,\ i = 1, \ldots, n$	Real-valued outcomes ordered so that $q_1 < q_2 < \cdots < q_n$.
$A = \{a_1, \ldots, a_m\}$	Set of m actions available to the agent.
$\pi_i(a)$	Probability of outcome q_i given action $a \in A$ is taken.
$B(a) = \sum_{i=1}^{n} \pi_i(a) q_i$	Expected gross benefit to the principal if action $a \in A$ is taken by the agent.
I_i	Payment to the agent when the outcome is q_i.
$U(a, I) = G(a) + K(a)V(I)$	Utility of the agent given action $a \in A$ and payment I.
$h(\cdot) = V^{-1}(\cdot)$	$V(\cdot)$ is assumed to be strictly increasing and strictly concave so that $h(\cdot)$ is well-defined.
$\overline{U}$	Agent's minimum utility.
$C_{FB}(a) \equiv h\left(\dfrac{\overline{U} - G(a)}{K(a)}\right)$	First-best cost associated with action $a \in A$; the agent's reservation price for selecting action a.
$C(a)$	Cost of implementing action $a \in A$.

We employ the same assumptions as G–H:

1. (a) $G(\cdot)$ and $K(\cdot)$ are real-valued, continuous functions defined on A, and K is positive; (b) for all $a_1, a_2 \in A$ and $I, \hat{I} \in \mathcal{I} \equiv (\underline{I}, \infty), G(a_1) + K(a_1)V(I) \geq G(a_2) + K(a_2)V(I) \Rightarrow G(a_1) + K(a_1)V(\hat{I}) \geq G(a_2) + K(a_2)V(\hat{I})$.
2. $[\overline{U} - G(a)]/K(a) \in \mathcal{U} \equiv \{v \mid v = V(I)\}$ for some $I \in \mathcal{I}$ for all $a \in A$.
3. For all $a \in A$ and $i = 1, \ldots, n$, $\pi_i(a) > 0$.
4. The principal is assumed to be risk neutral.

Assumption 1 implies that the agent's utility of income be risk independent of action. The functional form of the utility function is more general than the additively separable utility functions assumed in much of the agency literature, and admits additively and multiplicatively separable functions as special cases.

Assumption 3 eliminates the ability to infer actions from outcomes, thus establishing a "real" moral hazard problem.

Assumption 4 follows the preponderance of the analysis in G–H. Risk-neutrality of the principal avoids having to make utility comparisons between the principal and agent. It also permits the principal-agent problem into two parts. First, the principal determines the least (expected) cost way of implementing each of the m actions in A. An output-based incentive scheme $I_1, \ldots, I_n$ is said to ***implement*** action $a^* \in A$ if $I_1, \ldots, I_n$ solve the following mathematical programming problem:

Program 1:

$$\min_{I_1,\ldots,I_n} \sum_{i=1}^{n} \pi_i(a^*) I_i$$

s.t.

$$\sum_{i=1}^{n} \pi_i(a^*) U(a^*, I_i) \geq \overline{U} \qquad \text{(individual rationality)}$$

$$\sum_{i=1}^{n} \pi_i(a^*) U(a^*, I_i) \geq \sum_{i=1}^{n} \pi_i(a) U(a, I_i) \quad \text{for all } a \in A.$$

(incentive compatibility)

Given the incentive schemes for implementing each action, the second phase of the cost-benefit approach is entered. The principal chooses the action that maximizes his/her net expected payoff. The optimal action is a solution to the following mathematical program:

Program 2:

$$\max_{a \in A} \left(B(a) - \sum_{i=1}^{n} \pi_i(a) I_i \right).$$

If $a^* \in A$ is a solution to Program 2 and $I_1, \ldots, I_n$ implements action a^*, then $I_1, \ldots, I_n$ is called a ***second-best optimal incentive scheme.*** Our objective is to identify conditions under which a second-best optimal incentive scheme is nondecreasing in output, or equivalently (given the ordering on output), is nondecreasing in $i, i = 1, \ldots, n$.

Since $V(\cdot)$ is assumed to be both strictly increasing and strictly concave, $h(\cdot)$ is strictly increasing and strictly convex. Let $v_i = V(I_i)$, for $i = 1, \ldots, n$. Given the assumptions on

$U(\cdot,\cdot)$, an important feature of the cost-benefit approach is that Program 1, with decision variables $I_1,\ldots,I_n$, can be written as an equivalent mathematical program with decision variables $v_1,\ldots,v_n$. The new program has a convex objective function and a finite number of constraints that are each linear in the v_i's. To denote the explicit dependence of Program 1 on $a^* \in A$, we write the equivalent program as:

Program $1(a^*)$:

$$C(a^*) = \min_{v_1,\ldots,v_n} \sum_{i=1}^{n} \pi_i(a^*)h(v_i)$$

s.t.

$$G(a^*) + K(a^*)\sum_{i=1}^{n} \pi_i(a^*)v_i \geq \overline{U}$$

$$G(a^*) + K(a^*)\sum_{i=1}^{n} \pi_i(a^*)v_i \geq G(a) + K(a)\sum_{i=1}^{n} \pi_i(a)v_i \quad \text{for all } a \in A.$$

The Kuhn–Tucker conditions for Program $1(a^*)$ include the following:

(1) $$h'(v_i) = \left[\lambda + \sum_{\substack{a_j \in A \\ a_j \neq a^*}} \mu_j\right] K(a^*) - \sum_{\substack{a_j \in A \\ a_j \neq a^*}} \mu_j K(a_j)\left[\frac{\pi_i(a_j)}{\pi_i(a^*)}\right] \quad \text{for all } i,$$

where λ, $\mu_1,\ldots,\mu_m$ are nonnegative Lagrange multipliers.

3. MLRC AND THE G–H NUMERICAL EXAMPLE

There are two requirements for establishing the monotonicity of second-best optimal incentive schemes: the incentive scheme associated with some action must be nondecreasing in the state-descriptor *and* that action must be selected by the principal.

Suppose the X and Y are nonnegative random variables with densities f and g, respectively. These random variables are said to have the Monotone Likelihood Ratio (MLR) property if $f(x)/g(x)$ is nondecreasing in x. In the principal-agent context, MLRC holds if, for $a, a' \in A$, $a' \preceq a$ implies that $\pi_i(a')/\pi_i(a)$ is nonincreasing in i, where "$\preceq$" denotes a complete ordering on A defined as: $a' \preceq a$ if and only if $C_{FB}(a') \leq C_{FB}(a)$.

Since the state-descriptor i enters the optimality conditions (1) only through $\pi_i(a_j)/\pi_i(a^*)$, monotonicity of the incentive scheme that implements some action $a^* \in A$ follows if $\pi_i(a_j)/\pi_i(a^*)$ is decreasing in i for all actions a_j for which $\mu_j > 0$. If μ_j is positive for some action a_j that is "larger" than a^*, $\pi_i(a_j)/\pi_i(a^*)$ is increasing in i, thus confounding the demonstration that MLRC is sufficient for the monotonicity of second-best optimal sharing rules. In an attempt to demonstrate what can go wrong, G–H present the following numerical example:

$$n = 3,\ m = 3,\ V(I) = (3I)^{1/3},\ h(v) = \frac{1}{3}v^3,$$

$$\overline{U} = \frac{1}{4}\sqrt{2} + \frac{1}{12}\sqrt{\frac{7}{4}},\ \text{and } K(a) \equiv 1 \text{ for all } a.$$

Also

(2)

	$\pi_i(a_j)$ $i=1$	$i=2$	$i=3$	$G(a_j)$
a_1	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{12}$	0
a_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-\sqrt{2}}{12} - \frac{1}{4}\sqrt{\frac{7}{4}}$
a_3	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{-7}{12}\sqrt{\frac{7}{4}}$

It is clear from (2), that MLRC holds. The solution to Program $1(a_2)$ is:

$$v_1 = 0,\quad v_2 = \sqrt{2},\quad v_3 = \sqrt{\tfrac{7}{4}},\quad \lambda = 1.25,\quad \mu_1 = 2.0,\quad \mu_3 = 1.0,\quad \text{and } C(a_2) = 0.571.$$

Also, $C(a_1) = 0.0333$ and $C(a_3) = 0.7432$. Notice that $v_2 > v_3$ so the incentive scheme that implements a_2 is not monotone. Notice also, that $\mu_1 > 0$ and $\mu_3 > 0$, implying that the agent is indifferent between taking action a_2 and both actions a_1 and a_3. G–H (page 24) remark, "The reason monotonicity breaks down ... is because, at the optimum

[in Program $1(a_2)$], the agent is indifferent between a_2, the action to be implemented, a_1, a less costly action, and a_3, a more costly action."

The example is complete if it can be shown that it is optimal for the principal to choose action a_2. G–H state (page 24) that "... it is easy to show that we can find $q_1 < q_2 < q_3$, such that

$$B(a_2) - C(a_2) > \max\{B(a_3) - C(a_3),\ B(a_1) - C(a_1)\} \text{''} \tag{2a}$$

leading them to conclude that "... the optimal incentive scheme ... is not nondecreasing despite the satisfaction of MLRC." In the next section we demonstrate that this final claim is false.

4. A SUFFICIENT CONDITION

Within the context of the numerical example in the previous section, the problem of finding $q_1 < q_2 < q_3$ that guarantee that a_2 will be optimal for the principal is expressed as the following problem:

Find numbers q_1, q_2, q_3 that satisfy the following system of linear inequalities:

$$\begin{aligned} \sum_{i=1}^{3} \pi_i(a_2)q_i - C(a_2) &\geq \sum_{i=1}^{3} \pi_i(a_3)q_i - C(a_3) \\ \sum_{i=1}^{3} \pi_i(a_2)q_i - C(a_2) &\geq \sum_{i=1}^{3} \pi_i(a_1)q_i - C(a_1) \\ q_2 &\geq q_1 \\ q_3 &\geq q_2 . \end{aligned} \tag{3}$$

Notice that in (3) "$\geq$" replace "$>$" since any solution to (2a) necessarily satisfies (3).

Our first result identifies sufficient conditions for there not to be a solution to a particular system of linear inequalities. We then show that the conditions are satisfied in the G–H example.

Consider the following system of linear inequalities:

$$\begin{aligned} \sum_{i=1}^{3} \alpha_i q_i &\geq \Delta_1 \\ \sum_{i=1}^{3} \beta_i q_i &\geq \Delta_2 \\ q_2 - q_1 &\geq 0 \\ q_3 - q_2 &\geq 0 \\ q_i, \ i = 1, 2, 3 \quad &\text{unconstrained in sign.} \end{aligned} \tag{4}$$

We assume that the parameters $\Delta_1, \Delta_2, \alpha_i, \beta_i, i = 1, 2, 3$ satisfy:

$$\begin{gathered} \alpha_1 < \alpha_2 < \alpha_3, \quad \beta_1 > \beta_2 > \beta_3, \\ \alpha_1 < 0, \quad \alpha_3 > 0, \quad \beta_1 > 0, \quad \beta_3 < 0, \quad \Delta_2 < 0 \\ \text{and} \sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i = 0. \end{gathered} \tag{5}$$

PROPOSITION 1. *Suppose that (5) holds. If $\Delta_1/\Delta_2 < \min\{\alpha_1/\beta_1, \alpha_3/\beta_3\}$, then there is no solution to (4).*

PROOF: Using Gale's Theorem of the Alternative [see, e.g., Mangasarian (1969), page 33], (4) does not have a solution if there are variables y_1, y_2, y_3, y_4 that are each non-negative and that are all not zero, that satisfy the system of equations

$$A\mathbf{y} = \mathbf{b}, \tag{6}$$

where

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & -1 & 0 \\ \alpha_2 & \beta_2 & 1 & -1 \\ \alpha_3 & \beta_3 & 0 & 1 \\ \Delta_1 & \Delta_2 & 0 & 0 \end{pmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We will show that under the hypotheses, (6) always has a solution so that (4) does not.

Since $\sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i = 0$, the sum of the first three rows of A is zero, and the rank of A is less than four. It is straightforward to determine that the rank of A is three and that there are an infinite number of solutions to (6). The general form of the solution is:

$$\begin{aligned} y_1 &= t \\ y_2 &= (1 - \Delta_1 t)/\Delta_2 \\ y_3 &= [\beta_1 - (\Delta_1\beta_1 - \Delta_2\alpha_1)t]/\Delta_2 \\ y_4 &= [-\beta_3 + (\Delta_1\beta_3 - \Delta_2\alpha_3)t]/\Delta_2 \end{aligned}$$

where t is any real number.

Since $\beta_1 > 0$ and $\Delta_2 < 0$, y_3 can be positive only when $\Delta_1\beta_1 > \Delta_2\alpha_1$ or, equivalently, when

$$\frac{\Delta_1}{\Delta_2} < \frac{\alpha_1}{\beta_1}. \tag{7}$$

Similarly, y_4 can be positive only when

$$\frac{\Delta_1}{\Delta_2} < \frac{\alpha_3}{\beta_3}. \tag{8}$$

If (7) and (8) are both satisfied, then there is a non-negative solution to (6) with at least one $y_i > 0$ and there is no solution to (4). *Q.E.D.*

We now use Proposition 1 to prove that within the context of the G–H example, there is no solution to (3), implying that action a_2 will not be chosen by the principal. Let

$$\alpha_i = \pi_i(a_2) - \pi_i(a_1) \text{ and } \beta_i = \pi_i(a_2) - \pi_i(a_3),\ i = 1, 2, 3. \tag{9}$$

It is straightforward to show that MLRC implies that these α's and β's satisfy (5). Let

$$\Delta_1 = C(a_2) - C(a_1) \text{ and } \Delta_2 = C(a_2) - C(a_3). \tag{10}$$

Note that in the example, $\Delta_2 = 0.571 - 0.743 = -0.172 < 0$ and $\Delta_1 = 0.571 - 0.033 = 0.538 > 0$, as required. In the numerical example, $\alpha_1 / \beta_1 = (1/3 - 2/3)/(1/3 - 1/12) = -4/3$, $\alpha_3 / \beta_3 = (1/3 - 1/12)/(1/3 - 2/3) = -3/4$. Since $\Delta_1 / \Delta_2 = 0.538/ - 0.172 = -3.128 < -4/3 = \min\{\alpha_1 / \beta_1, \alpha_3 / \beta_3\}$, the principal would never pick action a_2 in the G–H example. We will now show that this example cannot be rectified by a more judicious choice of parameter values.

5. MONOTONICITY: SOME PRELIMINARY RESULTS

Denote the matrix of the probabilities of outcomes as functions of action as:

$$\pi_i(a_j)$$

(11)
$$\begin{array}{cccc} a_1 & p_1 & p_2 & p_3 \\ a_2 & p_4 & p_5 & p_6 \\ a_3 & p_7 & p_8 & p_9 \end{array}$$

MLRC implies the following relations hold:

(12)
$$\begin{array}{llll} \text{a.} & p_4 - p_1 < 0 & \text{e.} & p_1/p_4 > p_2/p_5 \\ \text{b.} & p_4 - p_7 < 0 & \text{f.} & p_2/p_5 > p_3/p_6 \\ \text{c.} & p_6 - p_3 < 0 & \text{g.} & p_4/p_7 > p_5/p_8 \\ \text{d.} & p_6 - p_9 < 0 & \text{h.} & p_5/p_8 > p_6/p_9 \end{array}$$

Without loss of generality, number the actions so that $a_1 \preceq a_2 \preceq a_3$; i.e., $C_{FB}(a_1) \leq C_{FB}(a_2) \leq C_{FB}(a_3)$. Since a_1 is the least-costly action, it follows that $C(a_1) = h[(\overline{U} - G(a_1)/K(a_1)] = C_{FB}(a_1)$. (Action a_1 can be implemented be setting $I_i = C_{FB}(a_1)$ for all i.)

It is easy to show that an optimal incentive scheme is monotone if $C(a_3) < C(a_2)$. The principal prefers action a_3 to action a_2, since MLRC implies that $B(a_i)$, the gross expected benefit accruing to the principal, is nondecreasing in i. MLRC also implies that $\pi_i(a_j)/\pi_i(a_3)$ is decreasing in i for $j = 1, 2$, and 3. Therefore, the incentive scheme that implements a_3 is monotonically increasing in output. Since the incentive scheme that implements action a_1, the only other action that could possibly be second-best optimal, is constant, it also is monotonic.

In light of the discussion above, if there is a non-monotone sharing rule, it is when a_2 is a second-best optimal action. From now on we will concentrate on the optimal incentive scheme that implements action a_2. Let $v^* \equiv (v_1^*, v_2^*, v_3^*)$ denote such an incentive scheme; i.e., v^* is a solution to Program $1(a_2)$.

The incentive scheme v^* is non-monotone if $h'(v_i^*) > h'(v_k^*)$ for $k > i$. Suppose $h'(v_1^*) > h'(v_2^*)$, which, in light of (1), implies that

(13)
$$\sum_{j=1,3} \mu_j K(a_j) \frac{\pi_1(a_j)}{\pi_1(a_2)} < \sum_{j=1,3} \mu_j K(a_j) \frac{\pi_2(a_j)}{\pi_2(a_2)}.$$

Let $\mu_j' = \mu_j K(a_j)$, $j = 1, 2, 3$. Using (11), write (13) as:

(14)
$$\mu_1' \frac{p_1}{p_2} + \mu_3' \frac{p_7}{p_4} < \mu_1' \frac{p_2}{p_5} + \mu_3' \frac{p_8}{p_5}.$$

The only other possible non-monotonicity is $h'(v_2^*) > h'(v_3^*)$, which can be written as:

$$(15) \qquad \mu_1' \frac{p_2}{p_5} + \mu_3' \frac{p_8}{p_5} < \mu_1' \frac{p_3}{p_6} + \mu_3' \frac{p_9}{p_6}.$$

In summary, the optimal sharing rule is non-monotone if, and only if, either (14) or (15) hold.

It will be convenient to write (14) in the following form:

$$\mu_1' \left[\frac{p_1}{p_4} - \frac{p_2}{p_5}\right] < \mu_3' \left[\frac{p_8}{p_5} - \frac{p_7}{p_4}\right].$$

The bracketed terms on the left and right side of the inequality are each positive by MLRC conditions (12e) and (12g), respectively. Therefore, (14) is equivalent to

$$(14') \qquad \frac{\mu_1'}{\mu_3'} < \frac{\dfrac{p_8}{p_5} - \dfrac{p_7}{p_4}}{\dfrac{p_1}{p_4} - \dfrac{p_2}{p_5}} \equiv \gamma$$

Similarly, (13) can be written as:

$$(15') \qquad \frac{\mu_1'}{\mu_3'} < \frac{\dfrac{p_9}{p_6} - \dfrac{p_8}{p_5}}{\dfrac{p_2}{p_5} - \dfrac{p_3}{p_6}} \equiv \delta$$

The ratios γ and δ are used to characterize the structure of the optimal sharing rule. Suppose that $\gamma < \delta$. Then $\mu_1'/\mu_3' < \gamma$ implies that $h'(v_1^*) > h'(v_2^*) > h'(v_3^*)$, which can never be optimal by Proposition 5 in G–H (which states that an optimal second-best incentive scheme cannot be nonincreasing in output for all states). If $\gamma < \mu_1'/\mu_2' < \delta$, then $h'(v_1^*) \leq h'(v_2^*)$ and $h'(v_2^*) \geq h'(v_3^*)$. Finally, if $\gamma \leq \mu_1'/\mu_2'$, then $h'(v_1^*) \leq h'(v_2^*) \leq h'(v_3^*)$. Analogous statements hold when $\delta > \gamma$. Note that since $h(\cdot)$ is strictly increasing, $h'(v_i^*) \leq (\geq)\, h'(v_k^*)$ is equivalent to $h(v_i^*) - h(v_k^*) \leq (\geq)\, 0$.

Let

$$(16) \qquad \begin{aligned} &x \equiv h(v_1^*) - h(v_2^*), && y \equiv h(v_3^*) - h(v_2^*), \\ &\underline{\theta} \equiv \min\{\gamma, \delta\}, && \overline{\theta} \equiv \max\{\gamma, \delta\}, \text{ and} \\ &r \equiv \mu_1'/\mu_3'. \end{aligned}$$

We summarize the discussion above in tabular form as follows:

(17)

	$r < \underline{\theta}$	$\underline{\theta} \leq r < \overline{\theta}$	$r \geq \overline{\theta}$
$\gamma < \delta$	NA$^+$	$x < 0, y < 0$	$x < 0, y > 0$
$\gamma \geq \delta$	NA$^+$	$x > 0, y > 0$	$x < 0, y > 0$

$^+$ "NA" means " ruled out by Proposition 5, G–H"

The next result provides an important linkage to Proposition 1, the sufficiency condition in Section 4. We use this result to reduce the number of cases that must be examined in the proof of the main result given in the next section.

PROPOSITION 2. *If* $\alpha_1/\beta_1 < (>)\, \alpha_3/\beta_3$, *then* $\gamma < (>)\, \delta$.

PROOF: Deferred to the Appendix.

We are now in position to state and prove the main result of this paper.

6. THE MAIN RESULT

In this section we prove that MLRC implies monotonicity of second-best optimal incentive schemes in the three-state cost-benefit model. The proof uses ideas developed in the previous sections: we show that if, for some action there is an incentive scheme that is not monotonic, then it will never be optimal for the principal to choose that action.

In light of the discussion in Section 5, non-monotonicity can only occur if $C(a_2) < C(a_3)$. Therefore, we assume that this condition holds and examine the case where

$$\Delta_1 > 0 \quad \text{and } \Delta_2 < 0, \tag{18}$$

with Δ_1 and Δ_2 as defined in (10).

We now state the main result of this paper.

THEOREM. *In the three-state, m–action cost-benefit principal-agent model, MLRC implies $I_1 \leq I_2 \leq I_3$, where I_1, I_2, I_3 is a second-best optimal incentive scheme.*

PROOF: Let $v^* = (v_1^*, v_2^*, v_3^*)$ and $w^* = (w_1^*, w_2^*, w_3^*)$ denote optimal solutions to Program $1(a_2)$ and Program $1(a_3)$, respectively.

Suppose that the incentive scheme that implements action a_2 is not monotonic.

CLAIM. *v^* is feasible in Program $1(a_3)$.*

PROOF OF CLAIM: The constraints of Program $1(a_2)$ are:

$$G(a_2) + K(a_2)\sum_{i=1}^{3}\pi_i(a_2)v_i \geq \overline{U} \tag{19}$$

$$G(a_2) + K(a_2)\sum_{i=1}^{3}\pi_i(a_2)v_i \geq G(a_1) + K(a_1)\sum_{i=1}^{3}\pi_i(a_1)v_i \tag{20}$$

$$G(a_2) + K(a_2)\sum_{i=1}^{3}\pi_i(a_2)v_i \geq G(a_3) + K(a_3)\sum_{i=1}^{3}\pi_i(a_3)v_i \tag{21}$$

and the constraints of Program $1(a_3)$ are:

$$G(a_3) + K(a_3)\sum_{i=1}^{3}\pi_i(a_3)w_i \geq \overline{U} \tag{22}$$

$$G(a_3) + K(a_3)\sum_{i=1}^{3}\pi_i(a_3)w_i \geq G(a_1) + K(a_1)\sum_{i=1}^{3}\pi_i(a_1)w_i \tag{23}$$

$$G(a_3) + K(a_3)\sum_{i=1}^{3}\pi_i(a_3)w_i \geq G(a_2) + K(a_2)\sum_{i=1}^{3}\pi_i(a_2)w_i. \tag{24}$$

Since v^* is not monotonic, it follows that (21) also holds as an equality when $v = v^*$. (MLRC implies that $\pi_i(a_1)/\pi_i(a_2)$ is decreasing in i and that $\pi_i(a_3)/\pi_i(a_2)$ is increasing in i. The only way in which v_i^* can not be monotonic is if μ_3, the coefficient of $\pi_i(a_3)/\pi_i(a_2)$ is positive, which, by complementary slackness, implies that (21) holds as an equality.) Since (21) holds as an equality, (24) is obviously satisfied, (22) follows from (19), and (23) follows from (22). *Q.E.D*

From (10),

$$
\begin{aligned}
\frac{\Delta_1}{\Delta_2} &= \frac{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - h[(\overline{U} - G(a_1))/K(a_1)]}{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - \sum_{i=1}^3 \pi_i(a_3)h(w_i^*)} \\
&< \frac{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - h[(\overline{U} - G(a_1))/K(a_1)]}{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - \sum_{i=1}^3 \pi_i(a_3)h(v_i^*)} \\
&< \frac{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - h(\sum_{i=1}^3 \pi_i(a_1)v_i^*)}{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - \sum_{i=1}^3 \pi_i(a_3)h(v_i^*)} \\
&< \frac{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - \sum_{i=1}^3 \pi_i(a_1)h(v_i^*)}{\sum_{i=1}^3 \pi_i(a_2)h(v_i^*) - \sum_{i=1}^3 \pi_i(a_3)h(v_i^*)} \\
&= \frac{\sum_{i=1}^3 \alpha_i h(v_i^*)}{\sum_{i=1}^3 \beta_i h(v_i^*)}
\end{aligned}
\tag{25}
$$

with α_i and β_i, $i = 1, 2, 3$ defined in (9). The first inequality follows since v^* is feasible in Program $1(a_2)$ and w^* is optimal in this program. Therefore, $\sum_{i=1}^3 \pi_i(a_2)h(w_i^*) < \sum_{i=1}^3 \pi_i(a_2)h(v_i^*)$. The second inequality follows since $C_{FB}(a_1) = C(a_1)$. The last inequality follows from the convexity of $h(\cdot)$.

Since $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 0$, we eliminate α_2 and β_2 in (25) to obtain

$$
\begin{aligned}
\frac{\Delta_1}{\Delta_2} &< \frac{\alpha_1 [h(v_1^*) - h(v_2^*)] + \alpha_3 [h(v_3^*) - h(v_2^*)]}{\beta_1 [h(v_1^*) - h(v_2^*)] + \beta_3 [h(v_3^*) - h(v_2^*)]} \\
&= \frac{\alpha_1 x + \alpha_3 y}{\beta_1 x + \beta_3 y} \equiv R,
\end{aligned}
\tag{26}
$$

where x and y are defined in (16). Proposition 2 asserts that there are only two ways in which v^* cannot be monotonic. We examine each case separately. (Recall that the α's and β's defined in (9) satisfy (5).)

Case 1: $\alpha_1/\beta_1 < \alpha_3/\beta_3$.

By Proposition 2, we know that in this case the only possible non-monotonicity is $x < 0$ and $y < 0$. Suppose $R > \alpha_1/\beta_1$. Then $(\beta_1\alpha_3 - \alpha_1\beta_3)y < 0$. Since $y < 0$, this implies that $\beta_1\alpha_3 - \alpha_1\beta_3 > 0$, which in turn implies that $\alpha_3/\beta_3 < \alpha_1/\beta_1$, a contradiction. Therefore, using (26), $\Delta_1/\Delta_2 < R < \min\{\alpha_1/\beta_1, \alpha_3/\beta_3\}$ and so by Proposition 1, (3) has no solution.

Case 2: $\alpha_1/\beta_1 > \alpha_3/\beta_3$.

Again using Proposition 2, the only possible non-monotonicity is $x > 0$ and $y > 0$. Suppose $R > \alpha_3/\beta_3$. Then $(\beta_3\alpha_1 - \alpha_3\beta_1) > 0$ and that $\beta_3\alpha_1 - \alpha_3\beta_1 > 0$. Since $x > 0$, this implies that $\alpha_1/\beta_1 < \alpha_3/\beta_3$, which again is a contradiction. As in Case 1, $\Delta_1/\Delta_2 < R < \min\{\alpha_1/\beta_1, \alpha_3/\beta_3\}$ and again invoking Proposition 1, (3) has no solution. *Q.E.D.*

The proof does not hinge on the fact that we only consider three actions. Suppose there are $m > 3$ actions. Let a^* denote some action for which there is a non-monotone incentive scheme. Let a_i denote an action that is "smaller" than a^* and has the property that the agent is indifferent between a_i and a^* in Program $1(a^*)$; if a^* is the least-costly action, the incentive scheme that implements a^* is constant with respect to i and hence is monotone. Let a_j denote any action that is "larger" than a_j^* and also has the property that the agent is indifferent between a_j and a^* in Program $1(a^*)$. If such an a_j does not exist, monotonicity follows directly. Otherwise, label the three actions as $a_1 = a_i$, $a_2 = a^*$, and $a_3 = a_j$. The theorem in this paper says that if the incentive scheme that implements a^* is not monotone, it is not optimal for the principal to select action a^*.

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APPENDIX

PROOF OF PROPOSITION 2: From (14') and (15'),

$$\operatorname{sgn}(\gamma - \delta) = \operatorname{sgn}\left[\frac{p_8/p_5 - p_7/p_4}{p_1/p_4 - p_2/p_5} - \frac{p_9/p_6 - p_8/p_5}{p_2/p_5 - p_3/p_6}\right]$$

$$= \operatorname{sgn}\left[\frac{p_4 p_8 - p_5 p_7}{p_1 p_5 - p_2 p_4} - \frac{p_5 p_9 - p_6 p_8}{p_2 p_6 - p_3 p_5}\right]. \tag{A1}$$

Substituting $p_2 = 1 - p_1 - p_3$, $p_5 = 1 - p_4 - p_6$, and $p_8 = 1 - p_7 - p_9$ into (A1) yields:

$$\operatorname{sgn}(\gamma - \delta) = \operatorname{sgn}\left[\frac{p_4(1 - p_7 - p_9) - p_7(1 - p_4 - p_6)}{p_1(1 - p_4 - p_6) - p_4(1 - p_1 - p_3)} - \frac{p_9(1 - p_4 - p_6) - p_6(1 - p_7 - p_9)}{p_6(1 - p_1 - p_3) - p_3(1 - p_4 - p_6)}\right]$$

$$= \operatorname{sgn}\left[\frac{p_4 - p_7 + p_6 p_7 - p_4 p_9}{p_1 - p_4 + p_3 p_4 - p_1 p_6} - \frac{p_9 - p_6 + p_6 p_7 - p_4 p_9}{p_6 - p_3 + p_3 p_4 - p_1 p_6}\right]. \tag{A2}$$

Let

(A3)

$a = p_9 - p_6 > 0$	by (12d)	$b = p_6 - p_3 > 0$	by (12c)
$c = p_4 p_9 - p_6 p_7 > 0$	by (12g) and (12h)	$d = p_1 p_6 - p_3 p_4 > 0$	by (12e) and (12f)
$e = p_4 - p_7 > 0$	by (12b)	$f = p_1 - p_4$	by (12a).

Note that in the derivation leading to (A2), $b - d = p_1 p_5 - p_2 p_4 > 0$ by (12e) and that $f - d = p_2 p_6 - p_3 p_5 > 0$ by (12f).

We use the following results.

LEMMA 1. *Suppose that a', b', c', d', e', and f' are positive numbers such that (i) $a'/b' > c'/d' > e'/f'$, (ii) $b' > d'$, and (iii) $f' > d'$. Then*

$$\frac{a' - c'}{b' - d'} > \frac{e' - c'}{f' - d'}. \tag{A4}$$

PROOF: Multiply both sides of $a'/b' > c'/d'$ by b'/c' to obtain $a'/c' > b'/d'$. Therefore, $(a' - c')/c' > (b' - d')d'$, which implies that $(a' - c')/(b' - d') > c'/d'$, since $b' - d' > 0$. Similarly, $c'/d' > e'/f'$ implies that $e'/f' > f'/d'$, so that $(e' - c')/c' < (f' - d')d'$. Therefore, $(e' - c')/(f' - d') < c'/d'$, since $f' - d' > 0$ and (A4) is obtained. *Q.E.D.*

LEMMA 2. *If a'', b'', c'', and d'' are positive numbers such that $a''/b'' > c''/d''$, then $a''/b'' > (a'' + c'')/(b'' + d'') > c''/d''$.*

PROOF: Similar to the proof of Lemma 1 and is omitted.

Note that with the definitions in (A3), (A2) becomes

$$\operatorname{sgn}(\gamma-\delta) = -\operatorname{sgn}\left[\frac{a-c}{b-d} - \frac{e-c}{f-d}\right].$$

The definition of the primed variables in Lemma 1 depend upon the sign of $(\alpha_1/\beta_1 - \alpha_3/\beta_3)$. We assume the following:

1. If $\alpha_1/\beta_1 < \alpha_3/\beta_3$, then $a' = a$, $b' = b, \cdots, f' = f$.
2. If $\alpha_1/\beta_1 > \alpha_3/\beta_3$, then $a' = e$, $b' = f$, $c' = c$, $d' = d$, $e' = a$, $f' = b$.

In the inequalities that follow, the first relation holds under Condition 1, while the parenthetical relation prevails under Condition 2.

Using (A3), hypothesis (i) of Lemma 1 is

$$\text{(A5)} \qquad \frac{p_9 - p_6}{p_6 - p_3} > (<) \frac{p_6 p_7 - p_4 p_9}{p_4 p_3 - p_1 p_6} > (<) \frac{p_4 - p_7}{p_1 - p_4}.$$

From the comment following (A3), hypotheses (ii) and (iii) of Lemma 1 are satisfied. If we show (A5) holds, Lemma 1 implies that $sgn(\gamma - \delta) = -sgn(\alpha_1/\beta_1 - \alpha_3/\beta_3)$, completing the proof.

The hypothesis that $\alpha_1/\beta_1 < (>)\ \alpha_3/\beta_3$ implies that

$$\frac{p_1(p_9 - p_6)}{p_1(p_6 - p_3)} > (<) \frac{p_3(p_4 - p_7)}{p_3(p_1 - p_4)}$$

or, invoking Lemma 2,

$$\text{(A6)} \qquad \frac{p_9 - p_6}{p_6 - p_3} > (<) \frac{p_1 p_9 - p_1 p_6 + p_3 p_4 - p_3 p_7}{p_1 p_6 - p_1 p_3 + p_1 p_3 - p_3 p_4}.$$

Now $(p_9 - p_6)/(p_6 - p_3) > (<)\ (p_4 - p_7)/(p_1 - p_4)$ implies that

$$\text{(A7)} \qquad p_1 p_9 - p_1 p_6 + p_3 p_4 - p_3 p_7 > (<)\ p_4 p_9 - p_6 p_7.$$

Substituting (A7) into (A6), yields

$$\frac{p_9 - p_6}{p_6 - p_3} > (<) \frac{p_4 p_9 - p_6 p_7}{p_1 p_6 - p_3 p_4} = \frac{p_6 p_7 - p_4 p_9}{p_3 p_4 - p_1 p_6}.$$

and the first inequality in (A5) is satisfied.

Using a similar argument, $\alpha_1/\beta_1 < (>)\ \alpha_3/\beta_3$ implies that

$$\frac{p_7(p_9 - p_6)}{p_7(p_6 - p_3)} > (<) \frac{p_9(p_4 - p_7)}{p_9(p_1 - p_4)}$$

or

$$\text{(A8)} \qquad \frac{p_7 p_9 - p_7 p_6}{p_7 p_8 - p_7 p_3} > (<) \frac{p_9 p_4 - p_9 p_7}{p_9 p_1 - p_9 p_4}.$$

Since the numerator and denominator in (A8) are both positive, it follows from Lemma 2 that

$$\text{(A9)} \qquad \frac{p_4 p_9 - p_6 p_7}{p_7 p_6 - p_7 p_3 + p_9 p_1 - p_9 p_4} > (<) \frac{p_9 p_4 - p_9 p_7}{p_9 p_1 - p_9 p_4} = \frac{p_4 - p_7}{p_1 - p_4}.$$

From (A7), we see that the denominator of the left-side of (A9) is greater than (less than) $p_1 p_6 - p_3 p_4$. Therefore,

$$\text{(A10)} \qquad \frac{p_4 p_9 - p_6 p_7}{p_1 p_6 - p_3 p_4} > (<) \frac{p_4 - p_7}{p_1 - p_4}.$$

Multiplying the top and bottom of left-side of (A10) yields the second inequality in (A5). *Q.E.D.*

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